

What Can We Learn from QED at Large Couplings?

A. W. Schreiber^a, R. Rosenfelder^b and C. Alexandrou^c

^a*Department of Physics and Mathematical Physics and Research
Centre for the Subatomic Structure of Matter
University of Adelaide, Adelaide, S. A. 5005, Australia*

^b*Paul Scherrer Institut, CH-5232 Villigen PSI, Switzerland*

^c*Department of Physics, University of Cyprus
CY-1678 Nicosia, Cyprus*

Abstract. In order to understand QCD at the energies relevant to hadronic physics one requires analytical methods for dealing with relativistic gauge field theories at large couplings. Strongly coupled quenched QED provides an ideal laboratory for the development of such techniques, in particular as many calculations suggest that – like QCD – this theory has a phase with broken chiral symmetry. In this talk we report on a nonperturbative variational calculation of the electron propagator within quenched QED and compare results to those obtained in other approaches. We find surprising differences among these results.

INTRODUCTION

It is well known that the content of a relativistic field theory (let us take QCD as an example) may be expressed in terms of functional averages of operators of the type

$$\int \mathcal{D}[\bar{\Psi}, \Psi, A] \mathcal{O}[\bar{\Psi}, \Psi, A] e^{-S_{\text{QCD}}[\bar{\Psi}, \Psi, A]} . \quad (1)$$

Even though it is not possible to perform these integrals exactly, we have learnt an awful lot over the years by studying Eq. (1) in various approximations. For example, if there is a small parameter (e.g. g , $1/N_c \dots$) one may develop well-ordered expansions of Eq. (1). Alternatively one can use stationary phase methods to approximate the integral – relevant, for example, if one is interested in elucidating the importance of classical configurations. Also, these methods are used in order

to gain understanding of the behaviour of very high orders of perturbation theory (HOPT). One may study the symmetries of the theory (e.g. chiral perturbation theory) in order to relate observables, or one can study its equations of motion (e.g. Dyson-Schwinger equation (DSE) studies). Finally, much progress has been made in recent years by actually evaluating Eq. (1) directly by discretizing it, as is done in Lattice QCD.

We have reported elsewhere on yet another technique, the so-called “worldline variational approach”, in which the path integral is approximated, in a rigorous and systematically correctable way, via a variational principle. This approach was initially applied to relativistic field theory in a scalar model [1] and more recently to quenched QED [2]. The common property that QED shares with QCD is that they are both renormalizable gauge field theories, in contrast to the scalar model which is super-renormalizable. Of course QCD, even after quenching (i.e. neglecting pair creation), differs from quenched QED by the fact that it is asymptotically free whereas quenched QED has a constant coupling. However, many of the above-mentioned approximative schemes for dealing with Eq. (1) should be applicable within either theory. Quenched QED therefore serves, and is often used, as a test-ground for these methods. In this talk we compare results for a particular quantity characterising quenched QED, namely its anomalous mass dimension, obtained via a) the variational approach, b) via perturbation theory, c) via Dyson-Schwinger equation studies and d) via HOPT. Surprisingly, we find that these results are not entirely consistent with each other, which indicates that our present understanding of these methods may be incomplete.

In Ref. [2] we derived from the variational approach an analytic, implicit, result for the anomalous mass dimension $\gamma_m(\alpha)$ of quenched, dimensionally regularized, QED within the MS scheme:

$$\alpha = \frac{4}{3} (1 + \gamma_m^{\text{var}}) \cot \frac{\pi/2}{1 + \gamma_m^{\text{var}}} . \quad (2)$$

This is a remarkable result. At small couplings it should (approximately) agree with the known result of perturbation theory¹. Indeed, as may be seen in Fig. 1 (taken from Ref. [2]), the variational result agrees with perturbation theory to within $\approx 20\%$ for couplings where 4th order perturbation theory appears applicable, i.e. for α less than about 1. We have also plotted an estimate of perturbation theory up to 5th order term of the perturbative expansion².

It is interesting to note that for a value of α around 1 the various orders of perturbation theory appear to ‘fan out’. It is tempting to speculate that this effect arises because of a finite radius of convergence of the perturbation expansion. This would be unexpected: the general wisdom from HOPT studies is that, as a

¹⁾ $\gamma_m(\alpha)$ has been calculated to fourth order in α within SU(N) [3]; by suitable choice of Casimirs (i.e. $C_A = 0$, $C_F = 1$ and, for a quenched theory, $N_f = 0$) the result for U(1) may be extracted from this. Recently, Broadhurst [4] has also calculated $\gamma_m(\alpha)$ to fourth order directly within QED

²⁾ The 5th order estimate has been derived using the Padé approximation methods of Ref. [5]

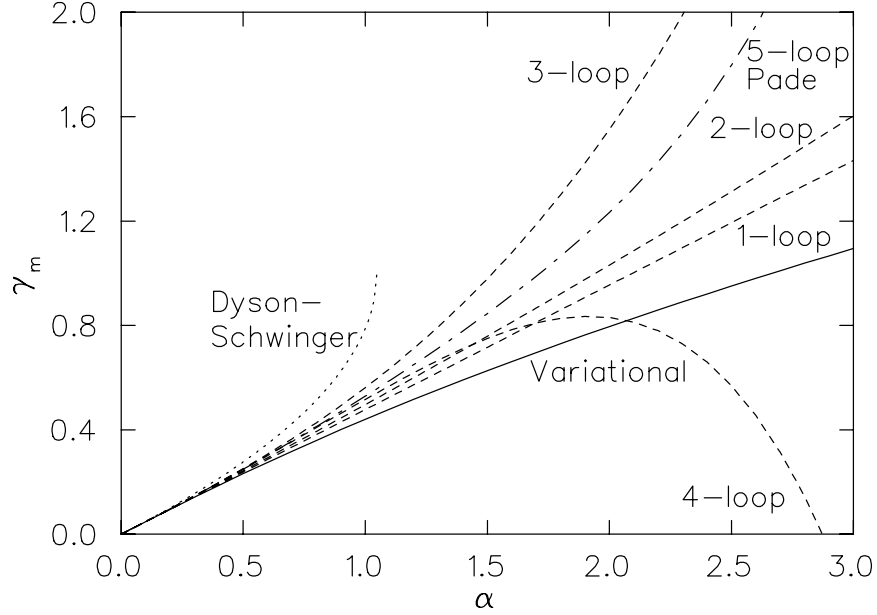


FIGURE 1. Anomalous mass dimension γ_m as function of the coupling constant α in quenched QED. The variational result (Eq. 2) is shown as a solid curve. The curves labeled “n-loop” show the result up to n-loop perturbation theory. The Padé estimation of the 5-loop result is shown as a dot-dashed line and finally, the solution from the Dyson-Schwinger equations in rainbow approximation is indicated as a dotted curve.

first rough estimate, the n^{th} order contribution to a perturbative expansion should scale roughly like the number of diagrams at that order. For the quenched QED propagator, the number of diagrams at order n is given by

$$N_n = \frac{(2n)!}{2^n n!} = (2n-1)!! \xrightarrow{n \rightarrow \infty} 2^{n+1/2} e^{-n} n^n \quad (3)$$

and hence the radius of convergence ($= \lim_{n \rightarrow \infty} |N_n|^{-1/n}$) of the perturbative expansion vanishes. More sophisticated analyses [6] confirm this general result (although, apparently, the high order behaviour of γ_m does not itself appear to ever have been calculated explicitly within quenched QED).

The final curve plotted in Fig. 1 is the prediction of a Dyson-Schwinger equation calculation within the rainbow approximation. This was obtained, for the dimensionally regularized theory within the MS scheme, in Ref. [2,7] and was found to be identical with the well known result for γ_m in the theory regularized with a hard cut-off, i.e. $\gamma_m^{\text{DS}} = 1 - \sqrt{1 - \frac{3}{\pi}\alpha}$. At small couplings this agrees with perturbation theory but then diverges from the latter in a region where (4^{th} order) perturbation theory still appears to be applicable. Above $\alpha = \pi/3$, γ_m^{DS} becomes complex, which is also the value α_{cr} of the coupling at which the famous chiral symmetry breaking of quenched QED occurs. Note also that the perturbative expansion of γ_m^{DS} has a

finite radius of convergence given by $\alpha_{con}^{DS} = \alpha_{cr}$. It is, however, not a great surprise that α_{con}^{DS} is finite in this calculation as the rainbow approximation contains exactly one diagram at each order and hence does not reproduce the factorial growth (Eq. 3) in the number of diagrams. Nevertheless, the coincidence of α_{con}^{DS} and α_{cr}^{DS} is interesting and one wonders if it will persist in Dyson-Schwinger calculations when going beyond the rainbow approximation. Also, if in the exact theory α_{con} actually vanishes one wonders what will happen to α_{cr} in that case.

The variational result plotted in Fig. 1 appears to show no sign of chiral symmetry breaking. Indeed it is easy to derive that at large couplings $\gamma_m^{var} \rightarrow \sqrt{3\pi/8} \sqrt{\alpha}$. However, it turns out that a perturbative expansion of γ_m^{var} also has a finite radius of convergence, not too different from the one mentioned above. We shall discuss this, as well as the large order behaviour of the perturbative expansion of γ_m^{var} , below. It is important to note that in this case a finite radius of convergence is not a trivial result of the approximation, as it was for rainbow QED: it can be shown [1] that each diagram, at any order in perturbation theory, is represented in some approximate way within the variational approximation; i.e. a perturbative expansion of γ_m^{var} would receive contributions from $(2n-1)!!$ diagrams at order n .

THE RADIUS OF CONVERGENCE OF γ_M^{VAR}

Equation (2) is an implicit equation for γ_m^{var} which can easily be solved numerically for arbitrary α . Importantly it can also be solved for complex α , so the determination of α_{con} amounts to a search for nontrivial analytic structure in the complex α plane. In Fig. 2 we show a plot of the (negative) real part of γ_m as a function of the (complex) coupling. Branch cuts, limiting the region of convergence of the perturbation expansion of γ_m^{var} , are clearly visible for negative $\mathcal{Re} \alpha$. Note that, as opposed to the Dyson-Schwinger equation result γ_m^{DS} , there are no cuts on the real axis (also for positive $\mathcal{Re} \alpha$, which is not shown in Fig. 2). Hence, in Fig. 1, γ_m^{var} is real for all α while γ_m^{DS} terminates at α_{con} [8].

The position of the cuts in Fig. 2 can be obtained straightforwardly by searching for the value of γ_m at which Eq. (2) has two distinct solutions infinitesimally close to each other; i.e. the value of γ_m^{var} at which the derivative with respect to γ_m^{var} of Eq. (2) vanishes. One obtains that the branchpoints are located at $\alpha = -0.496127 \pm 0.619172 i$, which yields a radius of convergence for the perturbation expansion of γ_m^{var} of $\alpha_{con}^{var} = 0.79342$.

THE BEHAVIOUR OF γ_M^{VAR} AT LARGE ORDERS IN PERTURBATION THEORY

The radius of convergence of the perturbative expansion of the anomalous mass dimension can also be obtained, of course, by directly examining the behaviour of its expansion coefficients at large orders, i.e. if we expand

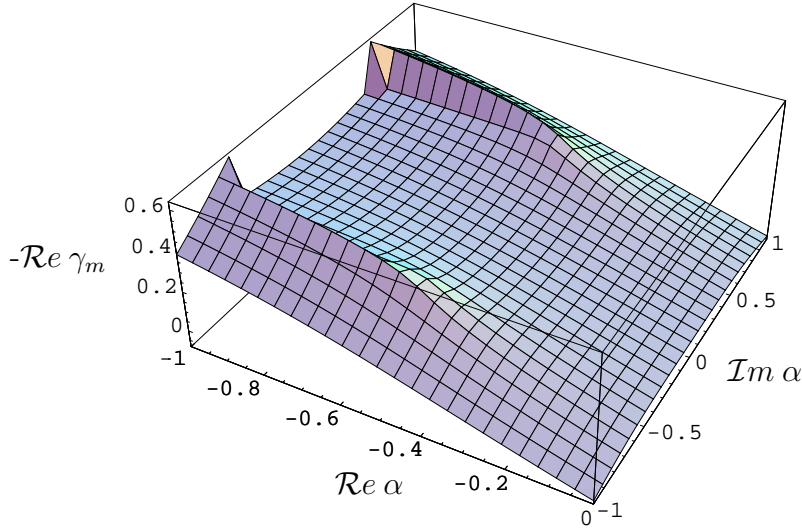


FIGURE 2. Solution of Eq. (2) for complex α . The figure was obtained by using Newton's method, with an initial seed value of $\gamma_m = 0$. No interesting analytic structure is found at positive $\text{Re } \alpha$, hence this region is not shown.

$$\gamma_m = \sum_{n=1}^{\infty} c_n \alpha^n \quad , \quad (4)$$

then radius of convergence is given by

$$\alpha_{con} = \lim_{n \rightarrow \infty} |c_n|^{-1/n} \quad . \quad (5)$$

For n up to ≈ 30 , it is possible to obtain these expansion coefficients by direct substitution of Eq. (4) into Eq. (2). The results are tabulated in Table 1, where it is clearly seen that although $|c_n|^{-1/n}$ does perhaps tend to a finite limit, the rate of convergence is rather slow (and non-uniform).

In order to find the c_n 's for higher values of n , it proves to be advantageous to convert Eq. (2) into a differential equation in order to eliminate the cotangent. We obtain

TABLE 1. Expansion coefficients of the perturbative expansion of γ_m^{var} .

| n | 5 | 10 | 15 | 20 | 25 | 30 |
|----------------|------|------|------|------|------|------|
| $ c_n ^{-1/n}$ | 2.00 | 1.31 | 1.16 | 1.11 | 1.02 | 1.16 |

$$1 + \gamma_m^{\text{var}} = \left(\alpha + \frac{2\pi}{3} \right) \gamma_m^{\text{var}}{}' - \frac{3\pi}{8} \alpha^2 \left(\frac{1}{1 + \gamma_m^{\text{var}}} \right)' . \quad (6)$$

If we now substitute Eq. (4) for γ_m^{var} , and $\sum_{n=0}^{\infty} a_n \alpha^n$ for $1/(1 + \gamma_m^{\text{var}})$ (hence $a_0 = 1$ and $a_n = -\sum_{k=1}^{n-1} c_{n-k} a_k$) we may solve for the coefficients c_n and a_n in an iterative manner. The results are shown in Fig. 3.

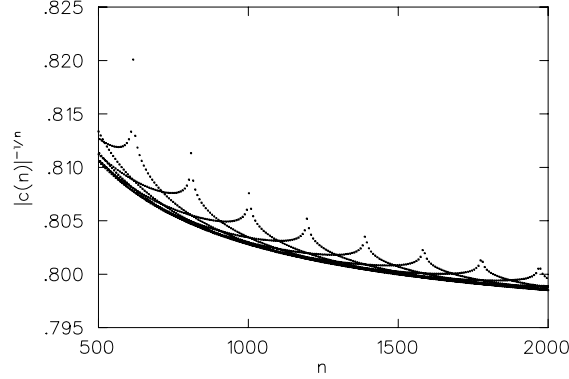


FIGURE 3. The coefficients c_n of the perturbative expansion of γ_m^{var} .

The slow rate of convergence, as well as its non-uniformity, is clearly visible in this figure. Indeed, numerically one finds that for large n the points in Fig. 3 are fitted exceedingly well by the functional form

$$c_n \approx (\alpha_{\text{con}}^{\text{var}})^{-n} \frac{e^{-\beta}}{n^{3/2}} \sin \left[\left(a + \frac{5\pi}{7} \right) n - \frac{3\pi}{7} + b \right] , \quad (7)$$

with $\beta \approx 1.376$, $a \approx 2.32 \times 10^{-3}$ and $b \approx -8.268 \times 10^{-2}$. These values of c_n may be compared to the equivalent expansion coefficients obtained from the Dyson-Schwinger equation result:

$$c_n^{\text{DS}} \approx \left(\alpha_{\text{con}}^{\text{DS}} \right)^{-n} \frac{e^{-1.27}}{n^{3/2}} . \quad (8)$$

In fact, the variational result and the rainbow Dyson-Schwinger result for the c_n 's are surprisingly similar, not only in their functional form but even in the numerical coefficients. The main difference is the occurrence of the sine function in the former. It is because of this sine that the branchcut, which for γ_m^{DS} lies on the real axis, has moved into the complex plane for γ_m^{var} .

CONCLUSION

In this contribution we have compared predictions for the anomalous mass dimension of quenched QED obtained through the use of a variety of techniques: the

worldline variational approach, perturbation theory up to $O[\alpha^4]$ (as well as a Padé estimate for the $O[\alpha^5]$ term), rainbow Dyson-Schwinger equation studies and general expectations from studies of high order perturbation theory. Both the rainbow DSE's and the variational approach yield a cut in γ_m as a function of the coupling. In the former this cut is on the real axis, and is associated with chiral symmetry breaking, while in the latter it has moved (a long way) off the real axis, so that in that calculation there is no obvious sign of chiral symmetry breaking. In either case, the cuts necessitate a finite radius of convergence for the perturbative expansion of $\gamma_m(\alpha)$, and one can argue that circumstantial evidence for this may be seen in the perturbative result as well. General expectations from HOPT, on the other hand, suggest that the radius of convergence should be zero. Clearly, as quenched QED is the prototype gauge theory for the investigation of chiral symmetry breaking, efforts should be made to clarify these disagreements. It may even be the case that the eventual resolution of these issues will teach us something about the limitations of one or more of these approximate techniques for dealing with nonperturbative gauge field theories.

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